A Model for the Sum of Two Squares

As Theorem 1.12 of "An Infinitude of Primes of the form b^2+1" establishes, if $4u^2+1$ is composite, then u must belong to the set of u such that

$u^2+t^2=n(n+1)$, where $t \le \dfrac{u^2-6}{5}$.

To help visualize the minimum requirement needed for an infinite sequence of u for which $4u^2+1$ is composite, we will introduce a simple model.
Our model will consist of a matrix of columns and rows, much like a modern day computer spread sheet with cells, rows and columns, in this case, each cell represents the sum of two squares. The column foot will simply be equal to the sum of two *identical* squares, and then proceed going up the column subtracting one from the first summand and squaring, adding one to the second summand and squaring until we get to a point where the subtracted summand side is 1. For example, suppose we start with the value of 5, then the column foot will simply be equal to the sum of two *identical* squares in this case: $5^2+5^2=50$, we then proceed going up the column, row by row following the rule, subtract one from the left summand and square, add one to the right summand and square, until the left summand reaches 1, as follows:

$$(5-4)^2+(5+4)^2=82 \quad \rightarrow \quad 1^2+9^2=82$$
$$(5-3)^2+(5+3)^2=68 \quad \rightarrow \quad 2^2+8^2=68$$
$$(5-2)^2+(5+2)^2=58 \quad \rightarrow \quad 3^2+7^2=58$$
$$(5-1)^2+(5+1)^2=52 \quad \rightarrow \quad 4^2+6^2=52$$
$$(5-0)^2+(5+0)^2=50 \quad \rightarrow \quad 5^2+5^2=50$$

Each column will be designated by the same value being squared on the first row, and to make algebraic representations easier, we will designate the first row as row zero.
The first seven columns of our model look like this:

						$1^2+13^2=170$	6
					$1^2+11^2=122$	$2^2+12^2=148$	5
				$1^2+9^2=82$	$2^2+10^2=104$	$3^2+11^2=130$	4
			$1^2+7^2=50$	$2^2+8^2=68$	$3^2+9^2=90$	$4^2+10^2=116$	3
		$1^2+5^2=26$	$2^2+6^2=40$	$3^2+7^2=58$	$4^2+8^2=80$	$5^2+9^2=106$	2
	$1^2+3^2=10$	$2^2+4^2=20$	$3^2+5^2=34$	$4^2+6^2=52$	$5^2+7^2=74$	$6^2+8^2=100$	1
$1^2+1^2=2$	$2^2+2^2=8$	$3^2+3^2=18$	$4^2+4^2=32$	$5^2+5^2=50$	$6^2+6^2=72$	$7^2+7^2=98$	0
1	2	3	4	5	6	7	

Figure 1

Now Theorem 1.12 proves that if $4u^2 + 1$ is composite, then u is a member of the set **S** such that $u^2 + t^2 = n(n+1)$, where $t \le \frac{u^2 - 6}{5}$. Hence our model needs to show the limit set by the condition $t \le \frac{u^2 - 6}{5}$. For purposes of simplification, we will always designate the left summand as u and the right summand as t. Since the purpose of this paper is to be expository, I will leave it as an exercise to the interested reader to find the equation for the maximum column height for each column established by the requirement that $t \le \frac{u^2 - 6}{5}$ (recall the starting value for each column always has u and t equal), and prove - this is a bit more difficult - that after $u = t = 25$, all elements of the set **S** are represented in our model.

Let us now examine how our model represents an element of the set **S**. We will examine $17^2 + 19^2 = 650$ where $650 = 25(25+1)$. Observe column 18, row 1 in the illustration below:

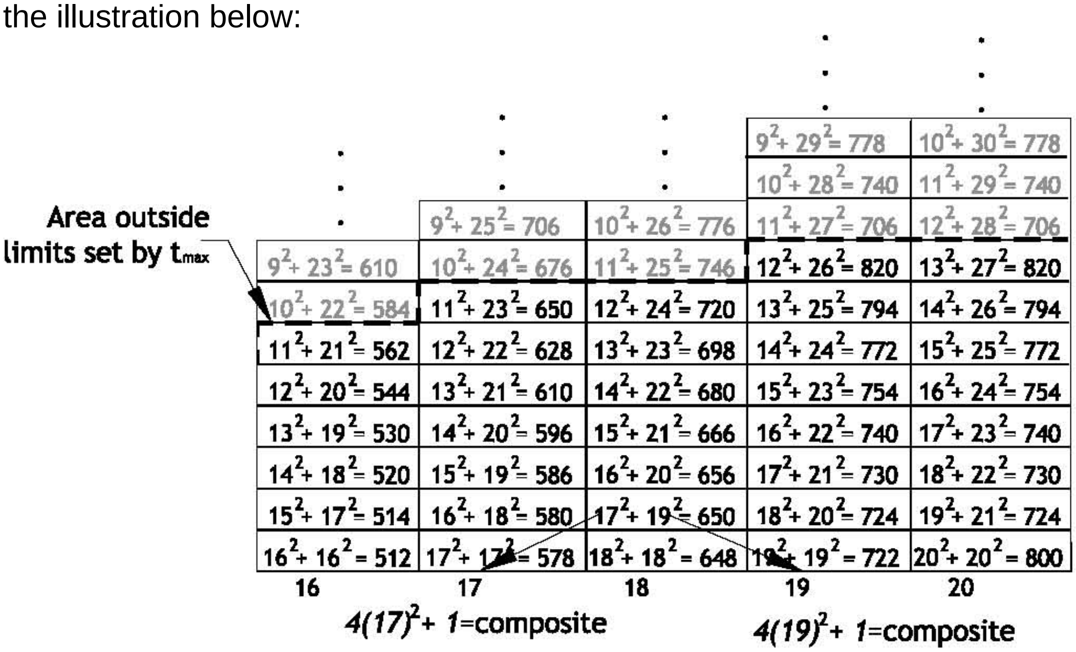

Figure 2

Since our model cuts off values of t greater than $t \le \frac{u^2 - 6}{5}$, each summand in our model results in a composite value for $4u^2 + 1$ and $4t^2 + 1$. In the above example, both $4(17)^2 + 1$ and $4(19)^2 + 1$ are composite (recall that by eliminating "cells" with values of t greater than that allowed by $t \le \frac{u^2 - 6}{5}$, we can use commutativity such that for a specific cell where $u^2 + t^2 = n(n+1)$, *both* $4u^2 + 1$ and $4t^2 + 1$ are composite). The alert reader will also notice that in column 17 row 6 we find another representation of 650 as the sum of two squares. In this case, both $4(11)^2+1$ and $4(23)^2+1$ are composite

One more example.
Consider $306=17(17+1)$, which can be represented as: 9^2+15^2, and can be found in column 12, row 3 in our model.

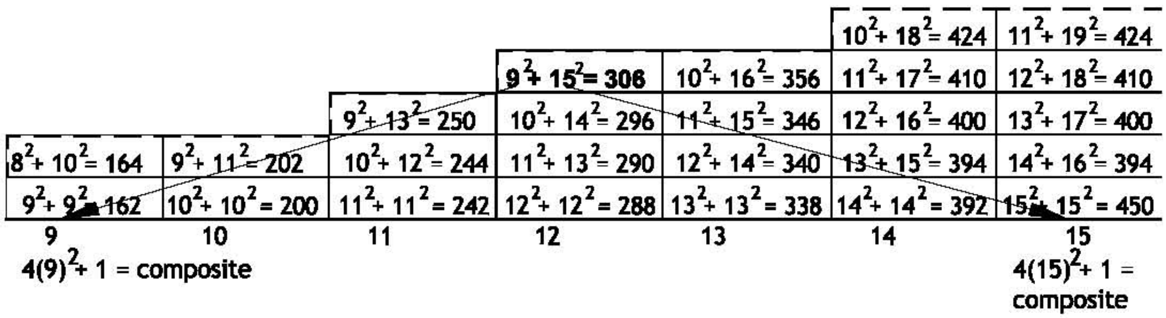

Figure 3

By Theorem 1.12, both $4(9)^2+1$ and $4(15)^2+1$ are composite.
Eliminating now the contents of the cells, and just examining the graphic result of this model, we have the following illustration:

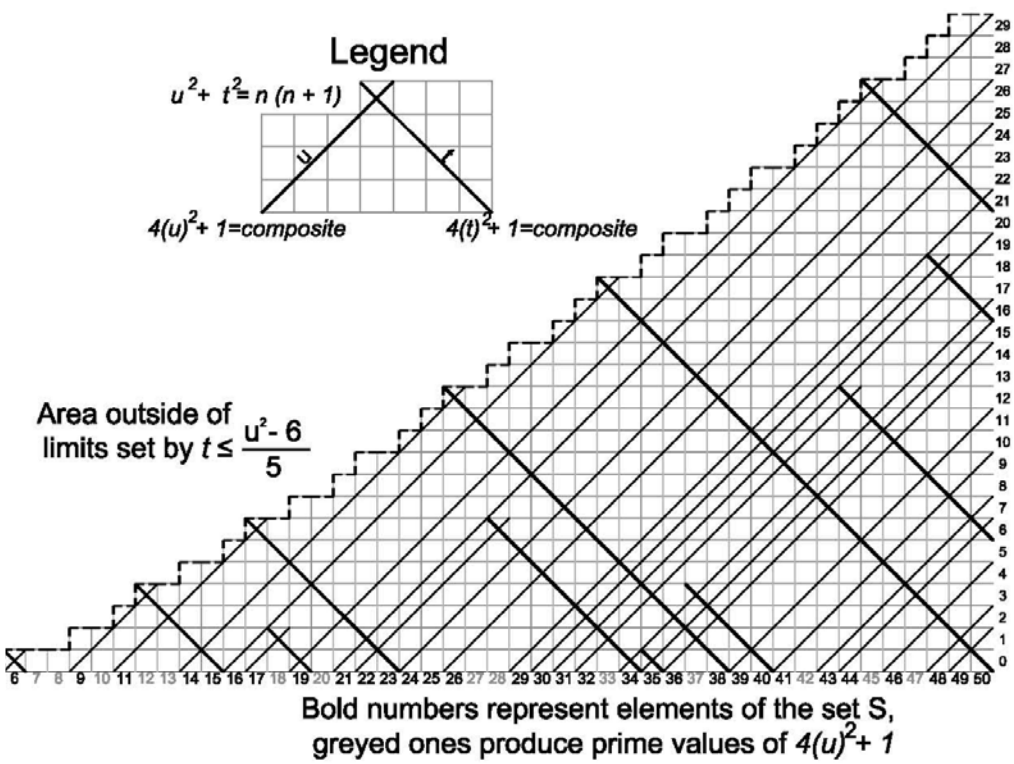

Figure 4

One nice visual aspect of this model is that each cell represents a distinct set of squared values, hence when a number has more than one representation as the sum of two squares, that representation will be in a different cell.
Moreover, we can glean from these two examples that a single representation of $u^2 + t^2 = n(n+1)$ produces *at most* two new elements to the set **S**. Figure 5 below shows why:

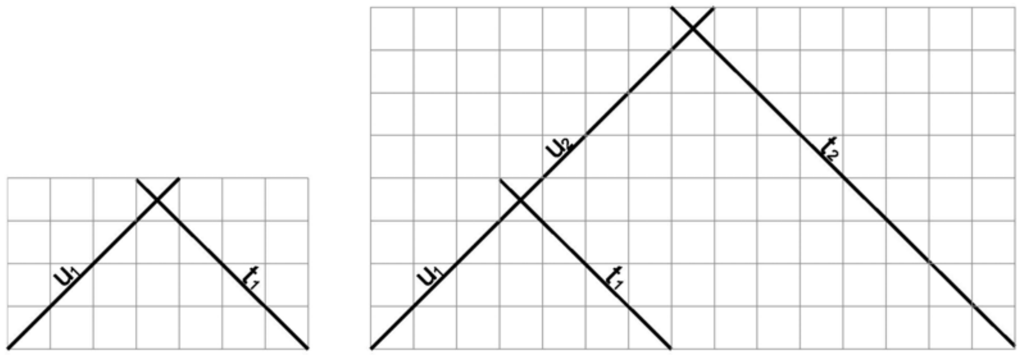

Figure 5

Since each cell represents a unique pair of squared terms, then either that cell "picks" out two unique values such that $4u^2 + 1$ and $4t^2 + 1$ are composite, or it shares a value with another cell (representation of $n(n+1)$ as the sum of two squares), in the second case above we have $u_1 = u_2$ and three composite values rather than four: $4(u_1)^2 + 1$, $4(t_1)^2 + 1$ and $4(t_2)^2 + 1$. The other possibility has $t_1 = t_2$.
It seems natural to call these figures triangles whose vertex lies in a cell such that $u^2 + t^2 = n(n+1)$, and whose left and right legs "pick" out elements of the set S along the number line at the bottom. Examining Figure 4 above, the vertex at column 12, row 3 has the left leg pick out 9 and the right leg pick out 15, and naturally it follows that $4(9)^2 + 1$ and $4(15)^2 + 1$ are both composite. Careful inspection shows that column 37 row 3 shares a left leg with another vertex not shown.

Now to have an infinite sequence of composites, according to our model, every number beyond some point on our number line must be "picked" by at least one leg. The bare minimum case would have one leg per each number on the number line, which means that there must be at least one representation of $n(n+1)$ as the sum of two squares per every 2 integers.